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## Classes of graphs with no long cycle as a vertex-minor are polynomially $\chi$ -bounded

Ringi Kim <sup>a,1</sup>, O-joung Kwon <sup>b,2</sup>, Sang-il Oum <sup>c,a,3</sup>,  
Vaidy Sivaraman <sup>d</sup>

<sup>a</sup> Department of Mathematical Sciences, KAIST, Daejeon, South Korea

<sup>b</sup> Department of Mathematics, Incheon National University, Incheon, South Korea

<sup>c</sup> Discrete Mathematics Group, Institute for Basic Science (IBS), Daejeon, South Korea

<sup>d</sup> Department of Mathematics, University of Central Florida, Orlando, United States of America

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### ABSTRACT

A class  $\mathcal{G}$  of graphs is  $\chi$ -bounded if there is a function  $f$  such that for every graph  $G \in \mathcal{G}$  and every induced subgraph  $H$  of  $G$ ,  $\chi(H) \leq f(\omega(H))$ . In addition, we say that  $\mathcal{G}$  is *polynomially  $\chi$ -bounded* if  $f$  can be taken as a polynomial function. We prove that for every integer  $n \geq 3$ , there exists a polynomial  $f$  such that  $\chi(G) \leq f(\omega(G))$  for all graphs with no vertex-minor isomorphic to the cycle graph  $C_n$ . To prove this, we show that if  $\mathcal{G}$  is polynomially  $\chi$ -bounded, then so is the closure of  $\mathcal{G}$  under taking the 1-join operation.

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E-mail addresses: [kimrg@kaist.ac.kr](mailto:kimrg@kaist.ac.kr) (R. Kim), [ojoungkwon@gmail.com](mailto:ojoungkwon@gmail.com) (O. Kwon), [sangil@ibs.re.kr](mailto:sangil@ibs.re.kr) (S. Oum), [vaidysivaraman@gmail.com](mailto:vaidysivaraman@gmail.com) (V. Sivaraman).

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## 1. Introduction

A class  $\mathcal{G}$  of graphs is said to be *hereditary* if for every  $G \in \mathcal{G}$ , every graph isomorphic to an induced subgraph of  $G$  belongs to  $\mathcal{G}$ . A class  $\mathcal{G}$  of graphs is  $\chi$ -*bounded* if there is a function  $f$  such that for every graph  $G \in \mathcal{G}$  and every induced subgraph  $H$  of  $G$ ,  $\chi(H) \leq f(\omega(H))$ . The function  $f$  is called a  $\chi$ -*bounding function*. This concept was first formulated by Gyárfás [9]. In particular, we say that  $\mathcal{G}$  is *polynomially  $\chi$ -bounded* if  $f$  can be taken as a polynomial function.

Recently, many open problems on  $\chi$ -boundedness have been resolved; see a recent survey by Scott and Seymour [14]. Yet we do not have much information on graph classes that are polynomially  $\chi$ -bounded. For instance, Gyárfás [9] showed that the class of  $P_n$ -free graphs is  $\chi$ -bounded but it is still open [8,13] whether it is polynomially  $\chi$ -bounded for  $n \geq 5$ . Regarding polynomially  $\chi$ -boundedness, Esperet proposed the following question, which remains open.

**Question 1.1** (Esperet; see [10]). *Is every  $\chi$ -bounded class of graphs polynomially  $\chi$ -bounded?*

Towards answering this question, it is interesting to know some graph operations that preserve the property of polynomial  $\chi$ -boundedness. If we have such graph operations, then we can use them to generate polynomially  $\chi$ -bounded graph classes.

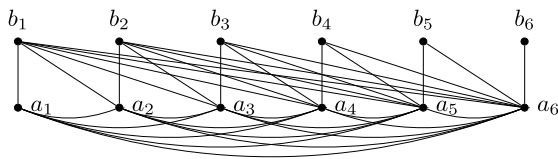
In this direction, Chudnovsky, Penev, Scott, and Trotignon [4] showed that if a hereditary class  $\mathcal{C}$  is polynomially  $\chi$ -bounded, then its closure under disjoint union and substitution is again polynomially  $\chi$ -bounded.

We prove the analog of their result for the 1-join. For graphs  $G_1$  and  $G_2$  with  $|V(G_1)|, |V(G_2)| \geq 3$  and  $V(G_1) \cap V(G_2) = \emptyset$ , we say that a graph  $G$  is obtained from  $G_1$  and  $G_2$  by 1-join if there are vertices  $v_1 \in V(G_1)$  and  $v_2 \in V(G_2)$  such that  $G$  is obtained from the disjoint union of  $G_1$  and  $G_2$  by deleting  $v_1$  and  $v_2$  and adding all edges between every neighbor of  $v_1$  in  $G_1$  and every neighbor of  $v_2$  in  $G_2$ . If so, then we say that  $G$  is the 1-join of  $(G_1, v_1)$  and  $(G_2, v_2)$ . For a class  $\mathcal{G}$ , let  $\mathcal{G}^\&$  be its closure under disjoint union and 1-join. Note that if  $\mathcal{G}$  is closed under taking isomorphisms, then so is  $\mathcal{G}^\&$ . We will see in Section 2 that  $\mathcal{G}^\&$  is hereditary if  $\mathcal{G}$  is hereditary.

**Theorem 1.2.** *If  $\mathcal{G}$  is a polynomially  $\chi$ -bounded class of graphs, then so is  $\mathcal{G}^\&$ .*

Dvořák and Král [7] and Kim [11] independently showed that for every hereditary class  $\mathcal{G}$  of graphs that is  $\chi$ -bounded, its closure under taking the 1-joins is again  $\chi$ -bounded. However, in both papers, the  $\chi$ -bounding function  $g$  for the new class is recursively defined as  $g(n) = O(f(n)g(n-1))$  for a  $\chi$ -bounding function  $f$  for  $\mathcal{G}$ . So,  $g(n)$  is exponential under their constructions.

We shall see that if  $f$  is a polynomial, then  $g(n-1)$  in the recurrence relation can be replaced by some polynomial  $f^*$ . This technique allows us to prove Theorem 1.2.

Fig. 1.  $K_6 \boxtimes \overline{K_6}$ .

As an application, we investigate the following conjecture of Geelen proposed in 2009. The definition of vertex-minors will be reviewed in Section 2.

**Conjecture 1.3** (Geelen; see [7]). *For every graph  $H$ , the class of graphs with no vertex-minor isomorphic to  $H$  is  $\chi$ -bounded.*

Conjecture 1.3 is known to be true when  $H$  is a wheel graph, shown by Choi, Kwon, Oum, and Wollan [1]. Motivated by the question of Esperet, we may ask the following.

**Question 1.4.** *Is it true that for every graph  $H$ , the class of graphs with no vertex-minor isomorphic to  $H$  is polynomially  $\chi$ -bounded?*

If this holds for  $H$ , then the class of  $H$ -vertex-minor free graphs satisfies the *Erdős-Hajnal property*, which means that there is a constant  $c > 0$  such that every graph  $G$  in this class has an independent set or a clique of size at least  $|V(G)|^c$ . Recently, Chudnovsky and Oum [3] proved that the Erdős-Hajnal property holds for the class of  $H$ -vertex-minor free graphs for all  $H$ .

We write  $P_n$  to denote the path graph on  $n$  vertices and  $C_n$  to denote the cycle graph on  $n$  vertices. Let  $K_n \boxtimes \overline{K_n}$  be the graph on  $2n$  vertices  $\{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}$  such that  $\{a_1, a_2, \dots, a_n\}$  is a clique,  $\{b_1, b_2, \dots, b_n\}$  is a stable set, and for all  $1 \leq i, j \leq n$ ,  $a_i$  is adjacent to  $b_j$  if and only if  $i \geq j$ . See Fig. 1 for an illustration of  $K_6 \boxtimes \overline{K_6}$ .

In Section 4, we prove the following theorem.

**Theorem 1.5.** *Let  $n \geq 4$ . If a graph  $G$  has no induced subgraph isomorphic to  $P_n$  or  $K_{\lceil n/2 \rceil} \boxtimes \overline{K_{\lceil n/2 \rceil}}$ , then*

$$\chi(G) \leq (n-3)^{\lceil n/2 \rceil - 1} \omega(G)^{\lceil n/2 \rceil - 1}.$$

We use this graph  $K_{\lceil n/2 \rceil} \boxtimes \overline{K_{\lceil n/2 \rceil}}$  because it has a vertex-minor isomorphic to  $P_n$ , shown by Kwon and Oum [12, Lemma 2.8].

**Lemma 1.6** (Kwon and Oum [12, Lemma 2.8]). *The graph  $K_{\lceil n/2 \rceil} \boxtimes \overline{K_{\lceil n/2 \rceil}}$  has a vertex-minor isomorphic to  $P_n$ .*

This allows us to obtain the following corollary of Theorem 1.5.

**Corollary 1.7.** *The class of graphs with no vertex-minor isomorphic to  $P_n$  is polynomially  $\chi$ -bounded.*

Kwon and Oum [12] proved the following theorem, stating that a prime graph with a long induced path must contain a long induced cycle as a vertex-minor. A graph is *prime* if it is not the 1-join of  $(G_1, v_1)$  and  $(G_2, v_2)$  for some graphs  $G_1, G_2$  with  $|V(G_1)|, |V(G_2)| \geq 3$ .

**Theorem 1.8** (Kwon and Oum [12]). *If a prime graph has an induced path of length  $\lceil 6.75n^7 \rceil$ , then it has a cycle of length  $n$  as a vertex-minor.*

We deduce the following stronger theorem from Corollary 1.7 by using Theorems 1.2 and 1.8. This answers Question 1.4 for a long cycle.

**Theorem 1.9.** *The class of graphs with no vertex-minor isomorphic to  $C_n$  is polynomially  $\chi$ -bounded.*

**Proof.** Let  $\mathcal{G}$  be the class of graphs having no vertex-minor isomorphic to  $P_m$  for  $m = \lceil 6.75n^7 \rceil$ . By Corollary 1.7,  $\mathcal{G}$  is polynomially  $\chi$ -bounded. By Theorem 1.2,  $\mathcal{G}^\&$  is polynomially  $\chi$ -bounded.

Let  $\mathcal{H}$  be the class of graphs having no vertex-minor isomorphic to  $C_n$ . Let  $G \in \mathcal{H}$ . We claim that  $G \in \mathcal{G}^\&$ . We may assume that  $G$  is connected. Every connected prime induced subgraph of  $G$  is in  $\mathcal{G}$  by Theorem 1.8. Then  $G$  can be obtained from connected prime induced subgraphs of  $G$  by taking 1-join repeatedly. Thus,  $G \in \mathcal{G}^\&$ . This proves that  $\mathcal{H} \subseteq \mathcal{G}^\&$ .  $\square$

Because  $C_m$  contains  $C_n$  as a vertex-minor whenever  $m \geq n$ , we may ask a stronger question on whether or not the class of graphs with no induced subgraph isomorphic to  $C_m$  for some  $m \geq n$  is polynomially  $\chi$ -bounded. It is not known. The following theorem of Chudnovsky, Scott, and Seymour [6] was initially a conjecture of Gyárfás [9] in 1985.

**Theorem 1.10** (Chudnovsky, Scott, and Seymour [6]). *The class of graphs with no induced subgraph isomorphic to a graph in  $\{C_m : m \geq n\}$  is  $\chi$ -bounded.*

We remark that as far as we know, it is not known whether the class of graphs with no  $P_5$  induced subgraph is polynomially  $\chi$ -bounded.

This paper is organized as follows. We will review necessary definitions in Section 2. In Section 3, we will present a proof of Theorem 1.2. In Section 4, we will prove Theorem 1.5. In Section 5, we discuss related problems.

## 2. Preliminaries

All graphs in this paper are simple and undirected. For a graph  $G$ , let  $V(G)$  and  $E(G)$  denote the vertex set and the edge set of  $G$ , respectively. A *clique* of a graph is a set of pairwise adjacent vertices. For a graph  $G$ , let  $\omega(G)$  be the maximum number of vertices in a clique of  $G$  and  $\chi(G)$  be the chromatic number of  $G$ .

Let  $G$  be a graph. For a vertex subset  $S$  of  $G$ , we denote by  $G[S]$  the subgraph of  $G$  induced by  $S$ . For a vertex  $v$  of  $G$ , we denote by  $G \setminus v$  the graph obtained from  $G$  by removing  $v$ . For an edge  $e$  of  $G$ , we write  $G \setminus e$  to denote the subgraph obtained from  $G$  by deleting  $e$ . For  $v \in V(G)$ , let  $N_G(v)$  be the set of neighbors of  $v$  in  $G$ . For a set  $X$  of vertices, let  $N_G(X) = (\bigcup_{v \in X} N_G(v)) \setminus X$ .

For two graphs  $G_1$  and  $G_2$ , the *disjoint union* of  $G_1$  and  $G_2$  is a graph  $(V(G'_1) \cup V(G'_2), E(G'_1) \cup E(G'_2))$  where  $G'_1$  is an isomorphic copy of  $G_1$  and  $G'_2$  is an isomorphic copy of  $G_2$  such that  $V(G'_1) \cap V(G'_2) = \emptyset$ . If  $V(G_1) \cap V(G_2) = \emptyset$ , then we take  $G'_1 = G_1$  and  $G'_2 = G_2$  for convenience.

For two graphs  $G_1$  and  $G_2$  on disjoint vertex sets and a vertex  $v \in V(G_1)$ , we say that a graph  $G$  is obtained from  $G_1$  by *substituting*  $G_2$  for  $v$  in  $G_1$ , if

- $V(G) = (V(G_1) \setminus \{v\}) \cup V(G_2)$ ,
- $E(G) = E(G_1 \setminus v) \cup E(G_2) \cup \{xy : x \in N_{G_1}(v), y \in V(G_2)\}$ .

For two sets  $A, A'$  with  $A \subseteq A'$ , we say that a function  $f' : A' \rightarrow B$  *extends* a function  $f : A \rightarrow B$  if  $f'(a) = f(a)$  for all  $a \in A$ .

**Lemma 2.1.** *If  $\mathcal{G}$  is a hereditary class of graphs, then so is  $\mathcal{G}^\&$ .*

**Proof.** We show that for every  $G \in \mathcal{G}^\&$  and  $v \in V(G)$ ,  $G \setminus v \in \mathcal{G}^\&$ . We proceed by induction on  $|V(G)|$ . If  $G \in \mathcal{G}$ , then we are done since  $\mathcal{G}$  is hereditary.

So, we may assume that  $G \notin \mathcal{G}$ . Then,  $G$  is the disjoint union of  $G_1$  and  $G_2$  for some  $G_1, G_2 \in \mathcal{G}^\&$  or the 1-join of  $(G_1, v_1)$  and  $(G_2, v_2)$  for some  $G_1, G_2 \in \mathcal{G}^\&$  and  $v_i \in V(G_i)$  for  $i = 1, 2$  where  $|V(G_1)|, |V(G_2)| \geq 3$ .

In the first case, we may assume that  $v \in V(G_1)$ . Then, by the induction hypothesis,  $G_1 \setminus v \in \mathcal{G}^\&$ , and since  $G \setminus v$  is the disjoint union of  $G_1 \setminus v$  and  $G_2$ , it follows that  $G \setminus v$  is contained in  $\mathcal{G}^\&$ .

In the second case, by symmetry, we may assume that  $v \in V(G_1) \setminus \{v_1\}$ . By the induction hypothesis,  $G_1 \setminus v \in \mathcal{G}^\&$ . If  $|V(G_1)| > 3$ , then  $G \setminus v$  is in  $\mathcal{G}^\&$  since it is the 1-join of  $(G_1 \setminus v, v_1)$  and  $(G_2, v_2)$ . If  $|V(G_1)| = 3$ , then  $G \setminus v$  is isomorphic to either  $G_2$  or the disjoint union of  $K_1$  and  $G_2 \setminus v_2$ . In either case  $G \setminus v$  is contained in  $\mathcal{G}^\&$ .  $\square$

*Vertex-minors* Now we are going to define vertex-minors. Actually, the readers may skip this part, if they assume Lemma 1.6 and Theorem 1.8 from other papers.

For a vertex  $v$  in a graph  $G$ , the *local complementation* at  $v$  results in the graph obtained from  $G$  by replacing the subgraph of  $G$  induced on  $N_G(v)$  by its complement. We write  $G * v$  to denote the graph obtained from  $G$  by applying local complementation at  $v$ . In other words,  $G * v$  is a graph on  $V(G)$  such that two distinct vertices  $x, y$  are adjacent in  $G * v$  if and only if exactly one of the following holds.

- (i) Both  $x$  and  $y$  are neighbors of  $v$  in  $G$ .
- (ii)  $x$  is adjacent to  $y$  in  $G$ .

A graph  $H$  is *locally equivalent* to  $G$  if  $H$  can be obtained from  $G$  by a sequence of local complementations. We say that a graph  $H$  is a *vertex-minor* of a graph  $G$  if  $H$  is an induced subgraph of a graph locally equivalent to  $G$ .

For an edge  $uv$  of a graph  $G$ , the *pivot* at  $uv$  is an operation to obtain  $G * v * u * v$  from  $G$ . We write  $G \wedge uv := G * u * v * u$ . We say that a graph  $H$  is a *pivot-minor* of a graph  $G$  if  $H$  is obtained from  $G$  by a sequence of pivots and vertex deletions.

### 3. Polynomially $\chi$ -boundedness for 1-join

For a class  $\mathcal{G}$  of graphs, let  $\mathcal{G}^*$  be the closure of  $\mathcal{G}$  under disjoint union and substitution. We will use the following result due to Chudnovsky, Penev, Scott, and Trotignon [4].

**Theorem 3.1** (Chudnovsky, Penev, Scott, and Trotignon [4]). *If  $\mathcal{G}$  is a polynomially  $\chi$ -bounded class of graphs, then  $\mathcal{G}^*$  is polynomially  $\chi$ -bounded.*

The following observation relates two graph classes  $\mathcal{G}^{\&}$  and  $\mathcal{G}^*$ .

**Lemma 3.2.** *Let  $\mathcal{G}$  be a hereditary class of graphs. If  $G \in \mathcal{G}^{\&}$  and  $v \in V(G)$ , then  $G[N_G(v)] \in \mathcal{G}^*$ .*

**Proof.** We prove by induction on  $|V(G)|$ .

If  $G \in \mathcal{G}$ , then we are done, since  $\mathcal{G}$  is closed under induced subgraphs and  $\mathcal{G} \subseteq \mathcal{G}^*$ . If  $G$  is the disjoint union of two graphs  $G_1, G_2$  from  $\mathcal{G}^{\&}$  and  $v \in V(G_1)$ , then by the induction hypothesis on the graph  $G_1$ , the claim follows.

Suppose that  $G$  is the 1-join of two graphs  $(G_1, v_1)$  and  $(G_2, v_2)$  where  $G_1, G_2 \in \mathcal{G}^{\&}$  and  $|V(G_1)|, |V(G_2)| \geq 3$ . Without loss of generality, we may assume that  $v \in V(G_1 \setminus v_1)$ . Let  $G_1[N_{G_1}(v)] = G_v$ , and  $G_2[N_{G_2}(v_2)] = G'_2$ . Then, by the induction hypothesis,  $G_v \in \mathcal{G}^*$ , and  $G'_2 \in \mathcal{G}^*$  because  $|V(G_1)|, |V(G_2)| < |V(G)|$ .

Since  $\mathcal{G}$  is hereditary, so is  $\mathcal{G}^*$ . We may assume that  $G'_2$  has at least one vertex because otherwise  $G[N_G(v)] = G_v \setminus v_1 \in \mathcal{G}^*$ . We may also assume that  $v$  is adjacent to  $v_1$  in  $G_1$  because otherwise  $G[N_G(v)] = G_v \in \mathcal{G}^*$ . Then  $G[N_G(v)]$  can be obtained from  $G_v$  by substituting  $G'_2$  for  $v_1$  and therefore  $G[N_G(v)]$  belongs to  $\mathcal{G}^*$ . This completes the proof.  $\square$

Let us now define a structure to describe how a connected graph in  $\mathcal{G}^\&$  is composed from graphs in  $\mathcal{G}$ . A *composition tree* is a triple  $(T, \phi, \psi)$  of a tree  $T$ , a map  $\phi$  defined on  $V(T)$  and a map  $\psi$  defined on  $E(T)$  such that

- for  $t \in V(T)$ ,  $\phi(t)$  is a connected graph, say  $G_t$ , on at least 3 vertices where graphs in  $\{G_t : t \in V(T)\}$  are vertex-disjoint,
- for  $st \in E(T)$ ,  $\psi(st) = \{u, v\}$  for some  $u \in V(G_s)$  and  $v \in V(G_t)$ , and
- for distinct  $e_1 \neq e_2 \in E(T)$ ,  $\psi(e_1)$  and  $\psi(e_2)$  are disjoint.

If a composition tree  $(T, \phi, \psi)$  is given, then one can construct a connected graph  $G$  from  $(T, \phi, \psi)$  by taking 1-joins repeatedly as follows:

- if  $|V(T)| = 1$ , say  $V(T) = \{t\}$ , then  $G = \phi(t)$ ,
- if  $|V(T)| > 1$ , let  $e = t_1 t_2 \in E(T)$  and  $T_i$  be the subtree of  $T \setminus e$  containing  $t_i$  for each  $i = 1, 2$ . Let  $\phi_i$  be the restriction of  $\phi$  on  $V(T_i)$  and  $\psi_i$  be the restriction of  $\psi$  on  $E(T_i)$  for each  $i = 1, 2$ . Let  $G_i$  be a graph constructed from  $(T_i, \phi_i, \psi_i)$  for  $i = 1, 2$ . Then,  $G$  is the 1-join of  $(G_1, v_1)$  and  $(G_2, v_2)$  where  $v_i \in V(G_i) \cap \psi(e)$ . It is straightforward to see that the choice of  $e$  does not make any difference to the obtained graph  $G$ .

If a vertex  $v$  of  $\phi(t)$  for some node  $t$  of  $T$  is in  $\psi(e)$  for some edge  $e$  of  $T$ , then  $v$  is called a *marker vertex*. After applying all 1-joins, marker vertices will disappear.

**Lemma 3.3.** *Let  $\mathcal{G}$  be a class of graphs. Let  $G$  be a connected graph in  $\mathcal{G}^\&$  with at least three vertices. Then there exists a composition tree  $(T, \phi, \psi)$  that constructs  $G$  such that  $\phi(t) \in \mathcal{G}$  for every node  $t$  of  $T$ .*

**Proof.** We proceed by induction on  $|V(G)|$ . We may assume that  $G \notin \mathcal{G}$ . Since  $G$  is connected,  $G$  is the 1-join of  $(G_1, v_1)$  and  $(G_2, v_2)$  for some graphs  $G_1, G_2$  in  $\mathcal{G}^\&$  and  $v_1 \in V(G_1), v_2 \in V(G_2)$  where  $|V(G_1)|, |V(G_2)| \geq 3$ . Since  $G$  is connected, both  $G_1$  and  $G_2$  are connected. By the induction hypothesis, we obtain two composition trees. We can combine them to obtain a composition tree  $(T, \phi, \psi)$  constructing  $G$ .  $\square$

**Lemma 3.4.** *Let  $c_1, c_2$  be positive integers. Let  $G$  be a connected graph constructed by a composition tree  $(T, \phi, \psi)$  such that  $\phi(t)$  is  $c_1$ -colorable for each node  $t$  of  $T$  and  $G[N_G(w)]$  is  $c_2$ -colorable for each vertex  $w$  of  $G$ . Let  $v$  be a vertex of  $G$ . Then for every proper  $c_2$ -coloring  $\beta$  of  $G[N_G(v)]$ , there exist functions  $\alpha' : V(G) \setminus \{v\} \rightarrow \{0, 1, 2, \dots, c_1\}$  and  $\beta' : V(G) \setminus \{v\} \rightarrow \{1, 2, \dots, c_2\}$  such that*

- (1)  $\alpha'(w) = 0$  and  $\beta'(w) = \beta(w)$  for every neighbor  $w$  of  $v$ ,
- (2)  $c = \alpha' \times \beta'$  is a proper  $(c_1 + 1)c_2$ -coloring of  $G \setminus v$ .

**Proof.** We proceed by induction on  $|V(G)|$ .

If  $|V(T)| = 1$ , then  $G = \phi(t)$  for the unique node  $t$  of  $T$  and so  $G$  has a proper  $c_1$ -coloring  $h : V(G \setminus v) \rightarrow \{1, 2, \dots, c_1\}$ . We define  $\alpha'$  and  $\beta'$  on  $V(G \setminus v)$  as follows:

$$\alpha'(w) = \begin{cases} 0 & \text{if } w \in N_G(v), \\ h(w) & \text{otherwise,} \end{cases} \quad \text{and} \quad \beta'(w) = \begin{cases} \beta(w) & \text{if } w \in N_G(v), \\ 1 & \text{otherwise.} \end{cases}$$

Clearly,  $\alpha' \times \beta'$  is a proper  $(c_1 + 1)c_2$ -coloring of  $G \setminus v$ .

Thus we may assume  $|V(T)| > 1$ . Let  $t_0$  be the unique node of  $T$  such that  $v \in V(\phi(t_0))$ . Let  $G_0 := \phi(t_0)$ . Let  $t_1, t_2, \dots, t_m$  be the neighbors of  $t_0$  in  $T$ . For each  $i \in \{1, 2, \dots, m\}$ , let  $v_i \in V(G_0)$  and  $u_i \in V(\phi(t_i))$  be vertices such that  $\psi(t_0 t_i) = \{v_i, u_i\}$  and let  $T_i$  be the connected component of  $T \setminus t_0$  containing  $t_i$ . For each  $i \in \{1, 2, \dots, m\}$ , let  $\phi_i$  be the restriction of  $\phi$  on  $V(T_i)$  and  $\psi_i$  be the restriction of  $\psi$  on  $E(T_i)$  and  $G_i$  be the graph constructed from a composition tree  $(T_i, \phi_i, \psi_i)$ .

Let  $h : V(G_0) \rightarrow \{1, 2, \dots, c_1\}$  be a proper  $c_1$ -coloring of  $G_0$ . Let  $\alpha'_0, \beta'_0$  be maps defined on  $V(G_0) \setminus \{v, v_1, \dots, v_m\}$  such that for  $w \in V(G_0) \setminus \{v, v_1, \dots, v_m\}$ ,

$$\alpha'_0(w) = \begin{cases} 0 & \text{if } w \in N_{G_0}(v), \\ h(w) & \text{otherwise,} \end{cases} \quad \text{and} \quad \beta'_0(w) = \begin{cases} \beta(w) & \text{if } w \in N_G(v), \\ 1 & \text{otherwise.} \end{cases}$$

Now we are going to define, for each  $i \in \{1, 2, \dots, m\}$ , a proper  $c_2$ -coloring  $\beta_i$  of  $G_i[N_{G_i}(u_i)]$ . If  $v_i$  is adjacent to  $v$  in  $G_0$ , then  $N_{G_i}(u_i)$  is a subset of  $N_G(v)$  and so let us define  $\beta_i$  to be the proper  $c_2$ -coloring of  $G_i[N_{G_i}(u_i)]$  induced by  $\beta$ .

If  $v_i$  is non-adjacent to  $v$  in  $G_0$ , then we claim that there exists a vertex  $y$  of  $G$  such that  $N_{G_i}(u_i) \subseteq N_G(y)$ . Since  $G_0$  is connected,  $v_i$  has a neighbor  $x$  in  $G_0$ . If  $x$  is not a marker vertex, then  $N_{G_i}(u_i) \subseteq N_G(x)$ . If  $x$  is a marker vertex, say  $x = v_j$  for some  $j \neq i$ , then there exists a neighbor  $y$  of  $u_j$  in  $G_j$  because  $G_j$  is connected and  $|V(G_j)| \neq 1$ . Now we observe that  $N_{G_i}(u_i) \subseteq N_G(y)$ . This proves the claim. By the claim, we can define  $\beta_i$  as a proper  $c_2$ -coloring of  $G_i[N_{G_i}(u_i)]$  induced by a proper  $c_2$ -coloring of  $G[N_G(y)]$ .

Observe that  $|V(G_i)| < |V(G)|$  for all  $i \in \{1, 2, \dots, m\}$  because  $G_0$  has at least three vertices. Now, by the induction hypothesis, for each  $i \in \{1, 2, \dots, m\}$ , there exist maps  $\alpha'_i : V(G_i \setminus u_i) \rightarrow \{0, 1, \dots, c_1\}$  and  $\beta'_i : V(G_i \setminus u_i) \rightarrow \{1, 2, \dots, c_2\}$  such that for every  $w \in N_{G_i}(u_i)$ ,

$$\alpha'_i(w) = \begin{cases} h(v_i) & \text{if } v_i \text{ is non-adjacent to } v \text{ in } G_0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta'_i(w) = \beta_i(w),$$

and  $\alpha'_i \times \beta'_i$  is a proper  $(c_1 + 1)c_2$ -coloring of  $G_i \setminus u_i$ , because we may swap colors 0 and  $h(v_i)$  in  $\alpha'_i$  after applying the induction hypothesis.



Now we define maps  $\alpha'$  and  $\beta'$  on  $V(G) \setminus \{v\}$  such that for  $w \in V(G) \setminus \{v\}$ ,

$$\alpha'(w) = \alpha'_i(w) \text{ and } \beta'(w) = \beta'_i(w) \text{ if } w \in V(G_i) \text{ for some } i \in \{0, 1, 2, \dots, m\}.$$

Clearly,  $\beta'$  extends  $\beta$ . In addition,  $\alpha'(w) = 0$  for all neighbors  $w$  of  $v$  in  $G$ .

We claim that  $c = \alpha' \times \beta'$  is a proper coloring of  $G \setminus v$ . Let  $x, y \in V(G \setminus v)$  be adjacent vertices in  $G \setminus v$ . If both  $x$  and  $y$  are neighbors of  $v$ , then  $\beta'(x) = \beta(x) \neq \beta(y) = \beta'(y)$ . So we may assume that  $y$  is not a neighbor of  $v$ .

- If  $x, y \in V(G_0)$ , then  $\alpha'(x) \neq \alpha'(y)$  because  $\alpha'(x) \in \{0, h(x)\}$  and  $\alpha'(y) = h(y) \neq 0$ .
- If  $x, y \in V(G_i)$  for some  $i \in \{1, 2, \dots, m\}$ , then  $(\alpha'(x), \beta'(x)) \neq (\alpha'(y), \beta'(y))$  because  $\alpha'_i \times \beta'_i$  is a proper coloring of  $G_i \setminus u_i$ .
- If  $V(G_0)$  contains exactly one of  $x$  and  $y$ , say  $x$ , then there exists  $i \in \{1, 2, \dots, m\}$  such that  $y \in V(G_i)$ . Then  $x$  is adjacent to  $v_i$  in  $G_0$  and  $y$  is adjacent to  $u_i$  in  $G_i$ . Since  $y$  is not adjacent to  $v$ ,  $v_i$  is not adjacent to  $v$  in  $G_0$  and so  $\alpha'(y) = \alpha'_i(y) = h(v_i)$ . As  $\alpha'(x) \in \{0, h(x)\}$ , we deduce that  $\alpha'(x) \neq \alpha'(y)$ .
- If  $x \in V(G_i)$  and  $y \in V(G_j)$  for distinct  $i, j \in \{1, 2, \dots, m\}$ , then  $x$  is adjacent to  $u_i$  in  $G_i$ ,  $v_i$  is adjacent to  $v_j$  in  $G_0$ , and  $u_j$  is adjacent to  $y$  in  $G_j$ . Since  $y$  is not adjacent to  $v$ ,  $v_j$  is not adjacent to  $v$  in  $G_0$  and so  $\alpha'(y) = \alpha'_j(y) = h(v_j)$ . Note that  $\alpha'(x) \in \{0, h(v_i)\}$  and therefore  $\alpha(x) \neq \alpha(y)$  because  $h$  is a proper coloring of  $G_0$ .

Therefore,  $c$  is a proper coloring of  $G \setminus v$ . This completes the proof.  $\square$

**Proof of Theorem 1.2.** We may assume that  $\mathcal{G}$  is hereditary, by replacing  $\mathcal{G}$  with the closure of  $\mathcal{G}$  under isomorphism and taking induced subgraphs, if necessary.

Let  $f$  be a  $\chi$ -bounding function for  $\mathcal{G}$  that is a polynomial. We may assume that  $1 \leq f(0) \leq f(1) \leq f(2) \leq \dots$ , by replacing  $f(x) = \sum_i a_i x^i$  with  $\sum_i |a_i| x^i$  if needed. By Theorem 3.1,  $\mathcal{G}^*$  is  $\chi$ -bounded by a polynomial  $f^*$ . We may assume that  $1 \leq f^*(0) \leq f^*(1) \leq f^*(2) \leq \dots$ .

We claim that

$$\chi(G) \leq (f(\omega(G)) + 1)f^*(\omega(G) - 1)$$

for all  $G \in \mathcal{G}^{\&}$ . This claim implies the theorem because  $\mathcal{G}^{\&}$  is hereditary by Lemma 2.1.

Let  $k = \omega(G)$ . We may assume that  $k > 1$ . We may assume that  $G$  is connected because  $\mathcal{G}^{\&}$  is hereditary and both  $f$  and  $f^*$  are non-decreasing. We may assume that  $G$  has at least three vertices. By Lemma 3.3,  $G$  has a composition tree  $(T, \phi, \psi)$  with  $\phi(x) \in \mathcal{G}$  for every node  $x$  of  $T$ . Note that  $\omega(\phi(x)) \leq k$  because  $\phi(x)$  is isomorphic to an induced subgraph of  $G$  and therefore  $\chi(\phi(x)) \leq f(\omega(\phi(x))) \leq f(k)$ . For each vertex  $w \in V(G)$ ,  $\omega(G[N_G(w)]) \leq k - 1$  and  $G[N_G(w)]$  belongs to  $\mathcal{G}^*$  by Lemma 3.2, and so  $G[N_G(w)]$  is  $f^*(k - 1)$ -colorable. Let  $v$  be a vertex of  $G$ . By Lemma 3.4, there exists a proper  $(f(k) + 1)f^*(k - 1)$ -coloring  $c = \alpha' \times \beta'$  of  $G \setminus v$  such that  $\alpha'(w) = 0$  for every

neighbor  $w$  of  $v$ . Then we can easily extend this to a proper  $(f(k) + 1)f^*(k - 1)$ -coloring of  $G$  by taking  $\alpha'(v) \neq 0$ .  $\square$

We remark that the same method can also prove that if  $\mathcal{G}$  is a class of graphs having an exponential  $\chi$ -bounding function, then so is  $\mathcal{G}^{\&}$ . This is because Chudnovsky, Penev, Scott, and Trotignon [4] also prove an analogue of Theorem 3.1 for classes of graphs having exponential  $\chi$ -bounding functions.

#### 4. Graphs without $P_n$ or $K_{\lceil n/2 \rceil} \boxtimes \overline{K_{\lceil n/2 \rceil}}$ induced subgraphs

Now we are ready to prove Theorem 1.5, which states that the class of graphs with no induced subgraph isomorphic to  $P_n$  or  $K_{\lceil n/2 \rceil} \boxtimes \overline{K_{\lceil n/2 \rceil}}$  is polynomially  $\chi$ -bounded.

For this section, a *path* from  $v$  to  $w$  is a sequence  $v_0 v_1 \cdots v_\ell$  of distinct vertices such that  $v_0 = v$ ,  $v_\ell = w$ , and  $v_i$  is adjacent to  $v_{i-1}$  for all  $i = 1, 2, \dots, \ell$ . We say a path  $Q := w_0 w_1 \cdots w_m$  *extends* a path  $P := v_0 v_1 \cdots v_\ell$  if  $m \geq \ell$  and  $v_i = w_i$  for all  $i \in \{0, 1, \dots, \ell\}$ .

For an induced path  $P$  from  $v$  to  $w$  in  $G$ , we write  $\Omega(G, P)$  to denote  $G \setminus (V(P) \cup N_G(V(P \setminus w)))$ . A component of  $\Omega(G, P)$  is *attached to*  $P$  if it contains a neighbor of  $w$ . A component  $C$  of  $\Omega(G, P)$  is *d-good* if the neighbors of  $w$  in  $C$  induce a graph of chromatic number larger than  $d$ . We say  $C$  is *d-bad* if it is not  $d$ -good. We say  $P$  is *d-good* in  $G$  if  $\Omega(G, P)$  has a  $d$ -good component.

**Lemma 4.1.** *Let  $k, d$  be positive integers. Let  $G$  be a graph. If  $\omega(G) \leq k$  and  $\chi(G) > kd$ , then  $G$  has a path  $P$  of length 1 such that  $\Omega(G, P)$  has a component  $C$  attached to  $P$  and  $\chi(C) > d$ .*

**Proof.** We may assume that  $G$  is connected. Let  $K$  be a maximum clique of  $G$ . By assumption,  $|K| \leq k$ . For each vertex  $x$  of  $K$ , let  $H_x = G \setminus N_G(x)$ . Since  $K$  is a maximum clique, for every vertex  $y$ , there is  $x \in K$  such that  $y \in V(H_x)$  and therefore  $\chi(G) \leq \sum_{x \in K} \chi(H_x)$ . So there exists  $x \in K$  such that  $\chi(H_x) > d$ . Let  $C'$  be a component of  $H_x$  such that  $\chi(C') = \chi(H_x)$ . Since  $\chi(C') > 1$  and  $x$  is an isolated vertex in  $H_x$ ,  $x \notin V(C')$ . Since  $G$  is connected, there is a shortest path  $v_0 v_1 \cdots v_\ell$  from  $x = v_0$  to some vertex  $v_\ell$  of  $C'$ . Let  $v := v_{\ell-2}$ ,  $w := v_{\ell-1}$ , and  $P := vw$ . Let  $C$  be the component of  $G \setminus N_G(v)$  containing  $C'$ . Then  $C$  is a component attached to  $P$  and  $\chi(C) > d$ .  $\square$

**Lemma 4.2.** *If a graph  $G$  has an induced path  $P$  of length at least 1 and  $\Omega(G, P)$  has a  $d$ -bad component  $C$  attached to  $P$  with  $\chi(C) > d$ , then there exist an induced path  $P'$  extending  $P$  by exactly 1 edge and a component  $C'$  of  $\Omega(G, P')$  attached to  $P'$  such that*

$$\chi(C') \geq \chi(C) - d.$$

**Proof.** Let  $w$  be the last vertex of  $P$ . Let  $C_w$  be the subgraph of  $C$  induced by the neighbors of  $w$ . Since  $C$  is  $d$ -bad,  $\chi(C_w) \leq d$  and therefore  $\chi(C \setminus N_G(w)) \geq \chi(C) - \chi(C_w) \geq \chi(C) - d > 0$ . So  $C \setminus N_G(w)$  has a component  $C'$  with  $\chi(C') \geq \chi(C) - d$ . Since

$C$  is connected, there is a vertex  $w' \in V(C_w)$  adjacent to some vertex in  $C'$ . We obtain  $P'$  by adding  $w'$  as a last vertex to  $P$ . Then  $C'$  is a component of  $\Omega(G, P')$  attached to  $P'$ .  $\square$

**Lemma 4.3.** *Let  $n \geq 4$ . Let  $G$  be a graph having no induced subgraph isomorphic to  $P_n$ . Let  $P$  be a path of length 1. If  $\Omega(G, P)$  has a component  $C$  attached to  $P$  with  $\chi(C) > d(n-3)$ , then  $G$  has a  $d$ -good induced path  $P'$  extending  $P$ .*

**Proof.** Suppose that  $G$  has no  $d$ -good induced path extending  $P$ . By applying Lemma 4.2  $(n-3)$  times, we can find an induced path  $P'$  of length  $n-2$  extending  $P$  and a component  $C'$  of  $\Omega(G, P')$  attached to  $P'$  such that  $\chi(C') \geq \chi(C) - d(n-3) > 0$ . We obtain an induced path of length  $n-1$  by taking  $P'$  and one vertex in  $C'$  adjacent to the last vertex of  $P'$ . This contradicts the assumption that  $G$  has no induced path on  $n$  vertices.  $\square$

Now we are ready to prove Theorem 1.5.

**Theorem 1.5.** *Let  $n \geq 4$ . If a graph  $G$  has no induced subgraph isomorphic to  $P_n$  or  $K_{\lceil n/2 \rceil} \square \overline{K_{\lceil n/2 \rceil}}$ , then*

$$\chi(G) \leq (n-3)^{\lceil n/2 \rceil - 1} \omega(G)^{\lceil n/2 \rceil - 1}.$$

**Proof.** Let  $k = \omega(G)$ . We may assume that  $G$  is connected. Let  $G_0 = G$ . Let  $d_i = (n-3)^{\lceil n/2 \rceil - i - 1} k^{\lceil n/2 \rceil - i - 1}$ . Note that  $\chi(G_0) > d_0$ .

Inductively we will find, in  $G_{i-1}$  of  $\chi(G_{i-1}) > d_{i-1}$ , an induced path  $Q_i$  and connected induced subgraphs  $C_i$ ,  $G_i$  of  $\chi(G_i) > d_i$  as follows. For  $i = 1, \dots, \lceil n/2 \rceil - 1$ , by Lemmas 4.1 and 4.3,  $G_{i-1}$  has a  $d_i$ -good induced path  $Q_i$  of length at least 1, because  $d_{i-1} = d_i k(n-3)$ . Let  $C_i$  be a  $d_i$ -good component of  $\Omega(G_{i-1}, Q_i)$  attached to  $Q_i$ . Among all components of the subgraph of  $C_i$  induced by the neighbors of the last vertex of  $Q_i$ , we choose a component  $G_i$  of the maximum chromatic number. By the definition of a  $d_i$ -good component,  $\chi(G_i) > d_i$ . This constructs  $G_1, G_2, \dots, G_{\lceil n/2 \rceil - 1}$ .

As  $\chi(G_{\lceil n/2 \rceil - 1}) > d_{\lceil n/2 \rceil - 1} = 1$ ,  $G_{\lceil n/2 \rceil - 1}$  contains at least one edge  $xy$ . By collecting the last two vertices of  $Q_1, Q_2, \dots, Q_{\lceil n/2 \rceil - 1}$  and  $x, y$ , we obtain an induced subgraph isomorphic to  $K_{\lceil n/2 \rceil} \square \overline{K_{\lceil n/2 \rceil}}$ , contradicting the assumption on  $G$ .  $\square$

## 5. Discussions

Corollary 1.7 and Theorem 1.5 show that

$$\chi(G) \leq (n-3)^{\lceil n/2 \rceil - 1} \omega(G)^{\lceil n/2 \rceil - 1}, \quad (1)$$

for a graph  $G$  with no vertex-minor isomorphic to  $P_n$ . It is natural to ask the following question.

**Question 5.1.** *Do there exist a constant  $c$  and a function  $f(n)$  such that*

$$\chi(G) \leq f(n)\omega(G)^c$$

*for all integers  $n$  and all graphs  $G$  with no vertex-minor isomorphic to  $P_n$ ?*

If we replace the vertex-minor condition of  $G$  in Question 5.1 with the condition having no induced subgraph isomorphic to  $P_n$  or  $K_{\lceil n/2 \rceil} \square \overline{K_{\lceil n/2 \rceil}}$ , then the answer is no, by an argument in [14]. We include its proof for the completeness.

**Proposition 5.2.** *For every constant  $c$  and a function  $f(n)$ , there exist a graph  $G$  and an integer  $n$  such that  $\chi(G) > f(n)\omega(G)^c$  and  $G$  has no induced subgraph isomorphic to  $P_n$  or  $K_{\lceil n/2 \rceil} \square \overline{K_{\lceil n/2 \rceil}}$ .*

**Proof.** There are known lower bounds on Ramsey numbers; for instance, Spencer [15] showed that for fixed  $k$ ,  $R(k, t) > t^{(k-1)/2+o(1)}$  for all sufficiently large  $t$ . It implies that for fixed  $k$ , for all sufficiently large  $t$ , there exists a graph  $G_t^{(k)}$  having no stable set of size  $k$  and  $\omega(G_t^{(k)}) < t$ ,  $|V(G_t^{(k)})| > t^{(k-1)/2+o(1)}$ . Then  $\chi(G_t^{(k)}) \geq t^{(k-1)/2+o(1)}/k$ .

Fix an integer  $k$  so that  $(k-1)/2 > c+1$ . Let  $c' := f(2k)$ . Then for all sufficiently large  $t$ ,  $\chi(G_t^{(k)}) > t^{(k-1)/2+o(1)}/k \geq c't^c$ . Then for all sufficiently large  $t$ ,  $\chi(G_t^{(k)}) > c'\omega(G)^c$  and  $G_t^{(k)}$  has no induced subgraph isomorphic to  $P_{2k}$  or  $K_k \square \overline{K_k}$ .  $\square$

The bound (1) cannot be improved for  $n = 4$ , as  $\chi(G) \geq \omega(G)$  in general. We will present the best possible bound for  $n = 5$ .

**Theorem 5.3.** *If a graph  $G$  has no vertex-minor isomorphic to  $P_5$ , then*

$$\chi(G) \leq \omega(G) + 1.$$

The following proposition trivially implies Theorem 5.3. We denote by  $W_n$  the wheel graph on  $n+1$  vertices.

**Proposition 5.4.** *Every graph with no vertex-minor isomorphic to  $P_5$  is perfect, unless it has a component isomorphic to  $C_5$  or  $W_5$ .*

In order to prove Proposition 5.4, we need to define the following graph classes. See Fig. 2 for an illustration.

- $W'_4$ : the graph obtained from  $W_4$  by deleting a spoke.
- Banner: the graph obtained from  $C_4$  by adding a pendant edge.
- Bull: the graph obtained from  $C_3$  by adding two pendant edges to distinct vertices of  $C_3$ .

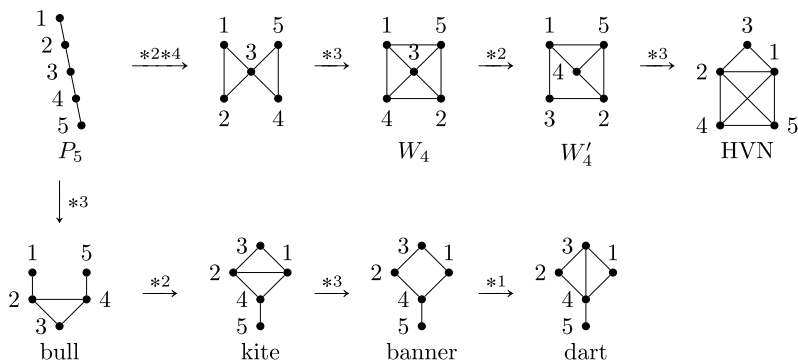


Fig. 2. Some graphs locally equivalent to  $P_5$ .

- Dart: the graph obtained from  $K_4 \setminus e$  for some edge  $e$  of  $K_4$  by adding a pendant edge to a vertex of degree 3.
- HVN: the graph obtained from  $K_4$  by adding a vertex of degree 2.
- Kite: the graph obtained from  $K_4 \setminus e$  for some edge  $e$  of  $K_4$  by adding a pendant edge to a divalent vertex.

We say that  $G$  is  $H$ -free if  $G$  has no induced subgraph isomorphic to  $H$ . We write  $\overline{G}$  to denote the complement of a graph  $G$ .

**Proof of Proposition 5.4.** From Fig. 2, it is easy to check that  $W_4$ ,  $W'_4$ , a banner, a bull, a dart, an HVN and a kite are locally equivalent to  $P_5$ . Therefore,  $G$  has no induced subgraph isomorphic to any of those graphs.

We may assume that  $G$  is connected. If  $G$  is  $C_5$ -free, then  $G$  does not contain an odd hole because  $G$  is  $P_5$ -free. Since  $\overline{C_5}$  is isomorphic to  $C_5$ ,  $G$  is  $\overline{C_5}$ -free. In addition, since  $\overline{W'_4}$  is the disjoint union of  $P_2$  and  $P_3$ ,  $G$  is  $\overline{C_k}$ -free for every odd  $k \geq 7$ . Therefore  $G$  is perfect by the strong perfect graph theorem [5].

Now we may assume that  $G$  contains  $C_5$  as an induced subgraph. Let  $L_i$  be the set of vertices of  $G$  having the distance  $i$  to  $C_5$ . We may assume that  $G$  is not  $C_5$ , that is,  $L_1$  is not empty.

We claim that  $L_1$  is complete to  $L_0$ . Suppose  $v \in L_1$  is not complete to  $L_0$ . Then  $v$  has exactly 1, 2, 3 or 4 neighbors in  $L_0$ . In each case it is easy to check that we can find an induced subgraph isomorphic to  $P_5$ , a bull, a banner or a kite, a contradiction.

Now we claim that  $L_2 = \emptyset$ . Suppose  $v \in L_2$ . Let  $u \in L_1$  such that  $uv$  is an edge. Now we see that  $G$  contains a dart, a contradiction.

If two vertices  $u, v$  in  $L_1$  are adjacent, then  $G$  contains a HVN as an induced subgraph, a contradiction. Thus,  $L_1$  is stable.

If  $L_1$  contains more than one vertex, then  $G$  contains  $W_4$ , a contradiction. So  $|L_1| = 1$ , and so  $G = W_5$ .  $\square$

Finally let us discuss pivot-minors instead of vertex-minors. We may ask the following question, which is stronger than Question 1.4.

**Question 5.5.** *Is it true that for every graph  $H$ , the class of graphs with no pivot-minor isomorphic to  $H$  is polynomially  $\chi$ -bounded?*

We will show that this is true if  $H$  is a path. Here is a useful lemma, replacing Lemma 1.6.

**Lemma 5.6.** *The graph  $K_n \boxtimes \overline{K_n}$  has a pivot-minor isomorphic to  $P_{n+1}$ .*

**Proof.** Let us remind that  $\{a_1, a_2, \dots, a_n\}$  is a clique and  $\{b_1, b_2, \dots, b_n\}$  is an independent set and  $a_i$  is adjacent to  $b_j$  if and only if  $i \geq j$ . It is easy to observe that  $(K_n \boxtimes \overline{K_n}) \wedge a_2 b_2 \wedge a_3 b_3 \wedge a_4 b_4 \wedge \dots \wedge a_{n-1} b_{n-1}$  has an induced path  $a_1 a_2 a_3 \dots a_n b_n$  of length  $n$ .  $\square$

So we deduce the following strengthening of Corollary 1.7 by the same proof.

**Proposition 5.7.** *The class of graphs with no pivot-minor isomorphic to  $P_n$  is polynomially  $\chi$ -bounded.*

However, the analogue of Theorem 1.8 is false for pivot-minors. As mentioned in [2], if  $k \not\equiv n \pmod{2}$ , then  $C_k$  has no pivot-minor isomorphic to  $C_n$ . This implies that there is a prime graph with an arbitrary long induced path having no pivot-minors isomorphic to  $C_n$ .

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